Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00042-9

J. Appl. Maths Mechs, Vol. 65, No. 3, pp. 371-376, 2001

0021-8928/01/\$—see front matter

THE RECONSTRUCTION OF UNBOUNDED CONTROLS IN NON-LINEAR DYNAMICAL SYSTEMS†

V. I. MAKSIMOV and L. PANDOLFI

Ekaterinburg and Turin (Italy)

e-mail: maksimov@imm.uran.ru; lucipan@polito.it

(*Received* 4 *May* 2000)

The problem of dynamical reconstruction of the variable input of a non-linear dynamical system, given the results of inaccurate observations of its phase trajectory, is considered. An algorithm for solving this problem is outlined, based on the method of control with a model. The algorithm is stable with respect to information noise and computation errors. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION. STATEMENT OF THE PROBLEM

Consider a dynamical system described by a non-linear differential equation

$$\dot{y}(t) = f_1(y(t)) + B_1u(t), \quad t \in [0, T], \quad y(0) = y_0; \quad y \in R^q \tag{1.1}$$

where B_1 is a $q \times n$ matrix and f_1 is a $q \times q$ matrix-valued function satisfying a Lipschitz condition. The trajectory $y(t)$ of the system depends on a time-varying input $u(t)$ (whose role is generally played by a control). Both the input and the trajectory are not given in advance. All that is known is that $u(t)$ is a square-summable function, that is,

$$u(\cdot) \in L_2([0, T]; R^n) \tag{1.2}$$

During the motion a certain signal characterizing the phase state of the system is observed. Namely, at discrete and sufficiently frequent instants of time $\tau_i \in [0, T]$ the quantities

$$x(\tau_i) = Cy(\tau_i) \in R^N$$

are observed with an error. The results of the measurements are vectors $\xi_i^h \in R^N$ such that

$$\xi_i^h = x(\tau_i) + z_i, \quad |z_i| \leq h$$

where C is a $q \times N$ matrix and the quantity h characterizes the accuracy of the measurements. It is required to construct an algorithm which is both dynamical and stable for the approximate reconstruction of the input. By "dynamical" we mean that the current values of the approximation to the input are produced in real time, while "stable" means that the approximation may be made as accurate as desired if the observation is sufficiently accurate.

In other words, the object of this paper is to construct an algorithm for the approximate retrieval (or, as is often said [1, 2], reconstruction) of the input-to calculate a certain control $v_h(\cdot) = \{v^h(\cdot), v^h(\cdot)\}$ such that the function $v^h(\cdot)$ plays the role of a kind of "estimate" for the approximation of $u(\cdot)$. We intend, first, to reconstruct, up to an arbitrary instant of time $t \in [0, T]$, the entire prehistory of the input $u(\tau)$, $0 \leq \tau \leq t$, using information that also pertains only to the prehistory of the process (measurements ξ_i^h corresponding to times $\tau_i < t$). Second, the process of reconstructing the input $u(\tau)$, $\tau \in [\tau_i, \tau_{i+1}]$ will be realized only after the control $v_h(t)$ has been computed in the previous time interval, that is, for $\tau \in [0, \tau]$. In so doing, we will also use the new information on the trajectory – the vector ξ_i^h.

The problem just formulated belongs to the class of inverse problems of dynamical estimation of unknown characteristics from the results of measurements. Such problems have been investigated (see, e.g., [3–6]). A dynamical algorithm was proposed in [7] to reconstruct the input $u(\cdot)$, given a closed

†*Prikl. Mat. Mekh.* Vol. 65, No. 3, pp. 385–391, 2001.

bounded set $P \subset R^n$ with the property

$$u(t) \in P \text{ for almost all } t \in [0, T]. \tag{1.3}$$

The algorithm is based on the method of positional control with a model, which is known from the theory of guaranteed control [8], combined will one of the main tools of the theory of ill-posed problems [9] – the smoothing functional method (Tikhonov's method). Later, the algorithm of [7] was developed for different classes of systems, described by (a) ordinary differential equations [1, 10], (b) equations with delay [11], and (c) parabolic and hyperbolic equations [2, 12–14]. In all the publications just cited, condition (1.3) was crucial.

In what follows, this algorithm will be modified in such a way as to permit reconstruction of an unbounded control – namely, a control for which it is known a priori that it satisfies condition (1.2). Incidentally, another method has been proposed [15, 16] for reconstructing unbounded controls (based on the so-called residue method [9]).

2. THE RECONSTRUCTION ALGORITHM

Before proceeding to describe the reconstruction algorithm, we will introduce a condition that will be assumed to hold throughout what follows.

Condition 1. A Lipschitzian function $f(z)$: $R^N \to R^N$ exists such that

$$Cf_1(y) = f(Cy)$$

for any $y \in R^q$.

To solve our problem, following the technique described previously in [1, 7], we choose a family of partitions

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = T, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h)$$

of the interval $[0, T]$ with diameters $\delta(h)$, and a function $\alpha(h)$ (the regularizer). The functions $\delta(h) \in (0, 1)$ and $\alpha(h) \in (0, 1)$ are chosen in such a way as to satisfy the following conditions

$$\delta(h) \to 0, \quad \alpha(h) \to 0, \quad \frac{h + \delta(h)}{\alpha(h)} \to 0, \quad \frac{h^2}{\delta(h)} \to 0 \quad \text{as} \quad h \to 0 \tag{2.1}$$

We then introduce an auxiliary control system (model)

$$\dot{w}(t) = f(\xi_i^h) + Bv^h(t) + \mathrm{v}^h(t), \quad t \in \delta_i = [\tau_i, \ \tau_{i+1}) \tag{2.2}$$

with initial condition $w(0) = \xi_0^h$, where $B = CB_1$.

Before the algorithm begins to run, we fix the quantity h and partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m}$. The run of the algorithm is divided into $m - 1$ steps of the same type. At the i-th step, performed in the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed. First, at time τ_i, vectors v_i^h and v_i^h are computed by the formulae

$$v_i^h = \frac{1}{\alpha} B'(\xi_i^h - w(\tau_i)), \quad \mathrm{v}_i^h = c\frac{\delta}{\alpha^2}(\xi_i^h - w(\tau_i)) \tag{2.3}$$

where $c = \text{const} > 0$ and the prime denotes transposition. Then the following controls are applied at the input of the model

$$v^h(t) = v_i^h \text{ and } \mathrm{v}^h(t) = \mathrm{v}_i^h, \quad t \in \delta_i \tag{2.4}$$

Under the action of these two controls, the model (2.2) is brought from the state $w(\tau_i)$ to the state $w(\tau_{i+1})$. The algorithm halts its run at time T.

We have the following

Theorem. Suppose conditions (2.1) are satisfied. Then

$$v^h(\cdot) \to u_*(\cdot) \text{ in } L_2([0, T]; R^n) \text{ as } h \to 0$$

where $u_*(\cdot) = u_*(\cdot; x(\cdot))$ is an element of the set $U(x(\cdot))$ of minimal $L_2([0, T]; R^n)$-norm and $U(x(\cdot))$ is the set of all controls $u(\cdot) \in L_2([0, T]; R^n)$ compatible with the output $x(\cdot)$.

Lemma 1. The set $U(x(\cdot))$ is convex anc closed in the space $L_2([0, T]; R^n)$.

By virtue of this lemma, the element $u_*(\cdot; x(\cdot))$ is uniquely defined.

The proof of the theorem follows a standard procedure (see, e.g., [1, 7, 12]) and is based on Lemma 2, which will be presented below. The proof of the lemma makes essential use of the Lipschitz property of a function f:

$$|f(x) - f(y)| \leq L|x - y|$$

which implies the following estimate for $t \in [\tau_i, \tau_{i+1})$

$$|f(x(t)) - f(\xi_i^h)| \leq k_0 Q, \quad k_0 = L\max\{1, |\dot{x}|_{L^2(0,T)}\}, \quad Q = h + \delta^{1/2} \tag{2.5}$$

It should also be noted that the proof that the algorithm converges is based on the stabilization procedure for a Lyapunov-type functional

$$\mu(t) = |w_h(t) - x(t)|^2 + \alpha\int_0^t [|v^h(s)|^2 - |u_*(s)|^2]ds \tag{2.6}$$

where $w_h(\cdot) = w(\cdot; \xi_0^h, v^h(\cdot))$ is the phase trajectory of model (2.2).

Lemma 2. Let

$$c > 8b^2, \quad 4\left(4bc + 2c^2 + \frac{3}{2} + \frac{5}{2}cb^3 + \frac{1}{2}c\right)\frac{\delta}{\alpha} \leq c, \quad \frac{\delta}{\alpha} \leq 1 \tag{2.7}$$

($b = |B|$ is the norm of the matrix B). Then one can find (explicitly) constants d_0 and d_1 such that

$$|x_i - w_i|^2 \leq d_0(h + \delta + \alpha), \quad x_i = x(\tau_i), \quad w_i = w_h(\tau_i) \tag{2.8}$$

$$\int_0^T |v^h(\tau)|^2 d\tau \leq \int_0^T |u_*(\tau)|^2 d\tau + d_1\frac{h+\delta}{\alpha} \tag{2.9}$$

Proof. Consider the quantity

$$\varepsilon(t) = \frac{1}{2}|x(t) - w_h(t)|^2$$

For almost all $t \in \delta_i = [\tau_i, \tau_{i+1})$, we have

$$\dot{\varepsilon}(t) = (x_i - w_i + \int_{\tau_i}^t \{f^i(\tau) + B^i(\tau) - \nu_i^h\}d\tau, \quad f^i(t) + B^i(t) - \nu_i^h)$$

$$(f^i(t) = f(x(t)) - f(\xi_i^h), \quad B^i(t) = B(u_*(t) - v_i^h))$$

Integrating the right- and left-hand sides of this equality, we see that for $t \in [\tau_i, \tau_{i+1})$,

$$\varepsilon(t) = \varepsilon_i + \sum_{j=0}^{7} M_i^j(t) + \nu_i^1(t) + \nu_i^2(t) - \int_{\tau_i}^t (x_i - w_i, \nu_i^h)d\tau$$

where

$$\varepsilon_i = \frac{1}{2}|x_i - w_i|^2,\quad M_i^j(t) = \int_{\tau_i}^t \mu_i^{(j)}(\tau)d\tau$$

$$\mu_i^{(0)}(t) = (\int_{\tau_i}^t B^i(\tau)d\tau,\ v_i^h),\ \mu_i^{(1)}(t) = (t-\tau_i)|v_i^h|^2$$

$$\mu_i^{(2)}(t) = (x_i - w_i,\ f^i(t)),\ \mu_i^3(t) = (\int_{\tau_i}^t f^i(\tau)d\tau,\ f^i(t))$$

$$\mu_i^{(4)}(t) = -(\int_{\tau_i}^t f^i(\tau)d\tau,\ v_i^h),\ \mu_i^{(5)}(t) = -\delta(v_i^h,\ f^i(t))$$

$$\mu_i^{(6)}(t) = -\delta(v_i^h,\ B^i(t)),\ \mu_i^{(7)}(t) = \int_{\tau_i}^t (x_i - w_i,\ B^i(\tau))d\tau$$

$$v_i^1(t) = 2(\int_{\tau_i}^t f^i(\tau)d\tau,\ \int_{\tau_i}^t B^i(\tau)d\tau),\ v_i^2(t) = |\int_{\tau_i}^t B^i(\tau)d\tau|^2;\ t\in\delta_i$$

It is obvious that

$$|v_i^h| = \left|\frac{B'(w_i - \xi_i^h)}{\alpha}\right| \le \frac{b}{\alpha}Q_i,\ |v_i^h| \le \frac{c\delta}{\alpha^2}Q_i;\ Q_i = h + \varepsilon_i^{1/2} \tag{2.10}$$

In addition, we have

$$M_i^0 \le b\left\{U_{i1} + \frac{\delta}{\alpha}Q_i\right\}\frac{c\delta^2}{\alpha}Q_i \le b\left(\frac{\delta}{c}U_{i2} + 4ch^2 + 4c\frac{\delta^3}{\alpha^3}\varepsilon_i\right) \tag{2.11}$$

$$U_{in} = \int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|^n\,d\tau$$

It follows from (2.10) that

$$M_i^1 \le c^2\frac{\delta^4}{\alpha^4}Q_i^2 \le 2c^2\left(h^2 + \frac{\delta^3}{\alpha^3}\varepsilon_i\right),\ t\in\delta_i \tag{2.12}$$

From (2.5), in turn, we obtain an estimate

$$M_i^2 \le k_0\varepsilon_i^{1/2}\delta Q \le \frac{c}{4}\ \frac{\delta^2}{\alpha^2}\varepsilon_i + k_*\alpha^2(h^2+\delta),\ t\in\delta_i,\ Q = h + \delta^{1/2} \tag{2.13}$$

It is not difficult to find a constant k_1 such that

$$M_i^3 \le k_1\delta^2 \tag{2.14}$$

Again using relations (2.10) and (2.5), we obtain

$$M_i^4 \le k_0 c\frac{\delta^3}{\alpha^2}Q_iQ \le k_0c\delta Qh + k_0^*\frac{\delta^3}{\alpha^2}\varepsilon_i^{1/2} \le$$

$$\le k_0^{**}\delta hQ + (k_0^*)^2\frac{\delta^3}{2\alpha} + \frac{\delta^3}{2\alpha^3}\varepsilon_i \le \frac{\delta^3}{2\alpha^3}\varepsilon_i + k_2\delta(h^2 + \delta + h\delta^{1/2}) \tag{2.15}$$

$$M_i^5 \le k_0c\frac{\delta^3}{\alpha^2}QQ_i \le \frac{\delta^3}{2a^3}\varepsilon_i + k_2\delta(h^2+\delta+h\delta^{1/2}) \tag{2.16}$$

After some algebraic manipulations, we obtain

$$M_i^6(t) \leq b^2 c \frac{\delta^2}{\alpha^2} Q_i \left(U_{i1} + \delta b \frac{Q_i}{\alpha} \right) \leq$$

$$\leq \frac{5}{2} c b^3 \frac{\delta^3}{\alpha^3} \varepsilon_i + 2cb^3 h^2 + cb^2 h U_{i1} + \frac{1}{2} c \delta b U_{i2} \tag{2.17}$$

$$M_i^7(\tau_{i+1}) \leq \int_{\tau_i}^{\tau_{i+1}} (\xi_i^h - w_i,\ B^i(\tau)) d\tau + bh \left(U_{i1} + b \frac{\delta Q_i}{\alpha} \right)$$

In addition

$$b^2 h \frac{\delta}{\alpha} \varepsilon_i^{1/2} \leq \frac{1}{2} b^4 h^2 + \frac{\delta^2}{2\alpha^2} \varepsilon_i$$

Then

$$M_i^7(\tau_{i+1}) \leq \int_{\tau_i}^{\tau_{i+1}} (\xi_i^h - w_i,\ B^i(\tau)) d\tau + bhU_{i1} + b^2 h^2 \frac{\delta}{\alpha} + \frac{1}{2} b^4 h^2 + \frac{\delta^2}{2\alpha^2} \varepsilon_i$$

It is obvious that

$$M_i^7(\tau_i + 1) \leq \int_{\tau_i}^{\tau_{i+1}} (\xi_i^h - w_i,\ B^i(\tau)) d\tau + bhU_{i1} + b^2 \left(1 + \frac{1}{2} b^2 \right) h^2 + \frac{\delta}{2\alpha} \varepsilon_i$$

The following inequalities follow from (2.10), (2.5) and the inequalities $\frac{\delta}{\alpha} \leq 1$, $\delta \leq 1$

$$\nu_i^1(t) \leq 2k_0 b \delta Q \left\{ b \frac{\delta}{\alpha} Q_i + U_{i1} \right\} \leq$$

$$\leq \frac{\delta^3}{2\alpha^3} \varepsilon_i + k_4 \{ \delta Q (Q + U_{i1}) \},\quad t \in \delta_i \tag{2.18}$$

$$\nu_i^2(t) \leq b^2 \left(U_{i1} + \delta b \frac{Q_i}{\alpha} \right)^2 \leq 4b^4 \frac{\delta^2}{\alpha^2} \varepsilon_i + 4b^4 h^2 + 2\delta b^2 U_{i2} \tag{2.19}$$

We then have

$$-\int_{\tau_i}^{\tau_{i+1}} (x_i - w_i,\ \nu_i^h) d\tau = -c \frac{\delta^2}{\alpha^2} (x_i - w_i,\ (\xi_i^h - w_i)) \leq$$

$$\leq -c \frac{\delta^2}{\alpha^2} \varepsilon_i + c \frac{\delta^3}{2\alpha^3} \varepsilon_i + \frac{1}{2} c \delta \alpha h^2 \tag{2.20}$$

We introduce the quantity

$$\mu_i = 2\varepsilon_i + \alpha \int_0^{\tau_i} \{ |\upsilon^h(\tau)|^2 - |u_*(\tau)|^2 \} d\tau$$

Combining relations (2.11)–(2.20), we obtain

$$\mu_{i+1} \leq d_1 \delta U_{i2} + d_2 (h + \delta^{3/2}) U_{i1} + d_3 (1 + \alpha^2 + \delta\alpha) h^2 + d_4 \delta^2 +$$

$$+ d_5 (\alpha^2 + h^2 + \delta) \delta + \mu_i + 2 \int_{\tau_i}^{\tau_{i+1}} (\xi_i^h - w_i,\ B^i(\tau)) d\tau + \alpha \int_{\tau_i}^{\tau_{i+1}} \{ |\upsilon_i^h|^2 - |u_*(\tau)|^2 \} d\tau +$$

$$+ \left(4bc + 2c^2 + \frac{3}{2} + \frac{5}{2} c b^3 + \frac{1}{2} c \right) \frac{\delta^3}{\alpha^3} \varepsilon_i + \left(4b^4 - \frac{3}{4} c \right) \frac{\delta^2}{\alpha^2} \varepsilon_i \tag{2.21}$$

In addition

$$|x(0)-\xi_0^h| \leq h \tag{2.22}$$

Inequalities (2.8) and (2.9) now follow from (2.7), (2.21) and (2.22).
The results of [1] and Lemma 2 imply the following lemma.

Lemma 3. Let $u_*(\cdot) = u_*(\cdot; x(\cdot))$ be a function of bounded variation. Then the rate of convergence of the algorithm satisfies the estimate

$$|v^h(\cdot)-u_*(\cdot)|^2_{L^2(0,T)} \leq c_1 \frac{h+\delta}{\alpha} + c_2(\alpha+\delta+h)^{1/2}$$

where c_1 and c_2 are constants which can be determined in explicit form.

Thus, if we put $\delta = \delta(h) \leq h$, $\alpha(h) = h^{2/3}$, we have the following estimate for the rate of convergence of the algorithm

$$|v^h(\cdot)-u_*(\cdot)|^2_{L^2(0,T)} \leq ch^{1/3}$$

This research was partially supported by the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS-96-0816), the Russian Foundation for Basic Research (98-01-00046) and the Programme for Supporting the Leading Scientific Schools of Russia (00-15-96086) for the first author; and the Italian MURST programme for the second author.

REFERENCES

1. KRYAZHIMSKII, A. V. and OSIPOV, Yu. S., *Inverse Problems for Ordinary Differential Equations: Dynamical Solutions*. Gordon and Breach, London, 1995.
2. OSIPOV, Yu. S., On the reconstruction of a parameter for hyperbolic system. *WP IIASA*, Laxenburg, Austria, 1991.
3. KURZHANSKII, A. B., *Control and Observation in Conditions of Uncertainty*. Nauka, Moscow, 1977.
4. GUSEV, M. I. and KURZHANSKII, A. B., Inverse problems of the dynamics of controllable systems. In *Mechanics and Scientific-Technological Progress*, Vol. 1: *General and Applied Mechanics*. Nauka, Moscow, 1987, 187–195.
5. CHERNOUS'KO, F. L., *Estimation of the Phase State of Dynamical Systems*. Nauka, Moscow, 1988.
6. OVSEYEVICH, A. I., TRUSHCHENKOV, V. L. and CHERNOUS'KO, F. L., Equations of continuous guaranteed estimation of the state of dynamical systems. *Izv. Akad. Nauk SSSR, Tekh. Kibern.*, 1984, 4, 94–101.
7. KRYAZHIMSKII, A. V. and OSIPOV, Yu. S., Modelling control in a dynamical system. *Izv. Akad. Nauk SSSR, Tekh. Kibern.*, 1983, 2, 51–60.
8. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
9. TIKHONOV, A. N. and ARSENIN, V. Ya., *Methods of Solving Ill-posed Problems*. Nauka, Moscow, 1979.
10. KRJAZIMSKII, A.N. and OSIPOV, Yu. S., On positional calculation of Ω-normal controls in dynamical systems. *Problems in Control and Information Theory*, 1984, **13**, 6, 425–436.
11. MAKSIMOV, V. I., Problems of robust control and dynamical input reconstruction for time-delay systems. *Proc. 5th Europ. Control Conf.*, Karlsruhe, Germany, 1999.
12. OSIPOV, Yu. S., KRYAZHIMSKII, A. V. and MAKSIMOV, V. I., *Problems of Dynamic Regularization for Systems with Distributed Parameters*. Inst. Mat. Mekh. Ural. Otd. Ross. Akad. Nauk, Sverdlovsk, 1991.
13. OSIPOV, Yu. S., PANDOLFI, L. and MAKSIMOV, V. I., Problems of dynamical reconstruction and robust boundary control: The case of Dirichlet boundary conditions. *Proc. International Conference "Distributed Systems: Optimization and Applications in Economics and Environmental Sciences"*. Ural. Otd. Ross. Akad. Nauk, Ekaterinburg, 2000, 42–45.
14. MAKSIMOV, V. I. and PANDOLFI, L., The reconstruction of controls in non-linear distributed systems. *Prikl. Mat. Mekh.*, 1999, **63**, 5, 716–724.
15. MAKSIMOV, V. I., Dynamical modelling of unknown perturbations in parabolic variational inequalities. *Prikl. Mat. Mekh.*, 1988, **52**, 5, 743–750.
16. MAKSIMOV, V. I., On the reconstruction of a control through results of observations. *Proc. 3rd Europ. Control Conf.*, Rome, Italy, 1995, 3766–3771.

Translated by D.L.